

Bagger-Witten line bundles on moduli spaces of elliptic curves

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In this paper we discuss Bagger-Witten line bundles over moduli spaces of SCFTs. We review how in general they are ‘fractional’ line bundles, not honest line bundles, twisted on triple overlaps. We discuss the special case of moduli spaces of elliptic curves in detail. There, the Bagger-Witten line bundle does not exist as an ordinary line bundle, but rather is necessarily fractional. As a fractional line bundle, it is nontrivial (though torsion) over the uncompactified moduli stack, and its restriction to the interior, excising corners with enhanced stabilizers, is also fractional. It becomes an honest line bundle on a moduli stack defined by a quotient of the upper half plane by a metaplectic group, rather than $SL(2, \mathbb{Z})$. We review and compare to results of recent work arguing that well-definedness of the world-sheet metric implies that the Bagger-Witten line bundle admits a flat connection (which includes torsion bundles as special cases), and give general arguments on the existence of universal structures on moduli spaces of SCFTs, in which superconformal deformation parameters are promoted to nondynamical fields ranging over the SCFT moduli space.

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1 Introduction

The Bagger-Witten line bundle was originally discovered as a structure over the moduli space of scalars in four-dimensional $N = 1$ supergravity theories [1], and was quickly demonstrated to exist as a structure over more general moduli spaces of two-dimensional SCFTs with $N = 2$ supersymmetry [2, 3], not only those with central charge $c = 9$ and integrally-charged chiral primaries.

Recently there have been a number of claims in the literature regarding torsion or triviality properties of the Bagger-Witten line bundle over either the uncompactified moduli space or the smooth part of the uncompactified moduli space, see *e.g.* [4, 5]. The purpose of this paper is to carefully examine these claims in the concrete example of moduli spaces of (complex structures on) elliptic curves, where a great deal is known in the mathematics literature. These moduli spaces are smooth (as stacks), and so are very amenable to discussion.

Briefly, we will encounter a few subtleties, but the results will essentially agree with the predictions of [5]. For example, in general the Bagger-Witten line bundle, despite the name, is not actually an honest line bundle, but rather is a ‘fractional’ line bundle, whose transition functions need not close on triple overlaps. In particular, this is true on moduli spaces of elliptic curves. As a result, even when we restrict to the interior of the moduli stack, away from points with enhanced stabilizers, it will not be possible in principle to trivialize the Bagger-Witten line bundle. Nevertheless, it is torsion (though nontrivial), and the restriction of its fourth tensor power to the interior of the moduli space is an honest and trivializable line bundle.

For another example, because the chiral spectral flow operator couples to the Bagger-Witten line bundle, it may not be possible to unambiguously define it over the entire moduli space. This is because fractional line bundles admit no global meromorphic sections. If the Bagger-Witten line bundle is fractional, then chiral spectral flow operators can be defined in local patches over the moduli space, but, there will be no way to globally stitch them together over the entire moduli space without encountering phase ambiguities.

In any event, these subtleties are consistent with the results of [5], as we shall review, as part of a more general discussion of universal structures over moduli spaces of SCFTs.

We begin by reviewing and extending known results on Bagger-Witten line bundles and Fayet-Iliopoulos parameters in four-dimensional $N = 1$ supergravity theories in section 2.1. In particular, we describe how the original Bagger-Witten story must be amended to accommodate ‘fractional’ line bundles, in which transition functions do not close on triple overlaps, to encompass standard examples. We then discuss worldsheet realizations of these results in section 2.2, and in particular observe that the Bagger-Witten story is generic in $N = 2$ SCFTs, not just those with $c = 9$ and integrally-charged chiral primaries.

In section 3 we focus on the particular example of moduli spaces of elliptic curves. These are well-understood in the mathematics literature, and so serve as a concrete example for discussions of Bagger-Witten. Over the uncompactified moduli space, the Bagger-Witten line bundle is a fractional line bundle, not an honest line bundle, which becomes an honest line bundle on a suitable gerbe¹ over the moduli space, one which can be presented as

$$[(\text{upper half plane}) / Mp(2, \mathbb{Z})],$$

for $Mp(2, \mathbb{Z})$ the metaplectic group extending $SL(2, \mathbb{Z})$ by \mathbb{Z}_2 . On that gerbe over the uncompactified moduli space, it is nontrivial (but torsion). If one restricts to the complement of the points with enhanced stabilizers, the fourth tensor power of the Bagger-Witten line bundle is trivializable. We also make a few remarks on higher-dimensional cases and applications of Bagger-Witten.

In section 4 we discuss and rephrase the recent computation of [5], which, by promoting superconformal deformation parameters to nondynamical fields and applying supersymmetry, argued that consistency of the worldsheet metric implies that the Bagger-Witten line bundle must admit a flat connection (which includes torsion bundles as special cases). We also compare this conclusion to results for Bagger-Witten line bundles on moduli spaces of elliptic curves.

In section 5, we make more general remarks on universal structures over moduli spaces of SCFTs. We do not claim to have a definition of a ‘universal SCFT,’ beyond a few intuitive ideas, but there are some general mathematical principles which can be applied, as we review. We also outline a proposal for how such a structure might be built more generally in terms of a section of a sheaf of renormalization group flows.

Finally, in section 6, we discuss other bundles over moduli spaces of SCFTs, analogous to the Bagger-Witten line bundle, that appear in *e.g.* heterotic string theories.

We will focus for most of this paper on two-dimensional SCFTs with (2,2) supersymmetry, with the exception of section 6. Theories with (0,2) supersymmetry have more complicated moduli spaces, as we shall discuss, and so are less amenable to discussion.

¹ Locally, a ‘gerbe’ is an orbifold by a finite trivially-acting group. Sigma models on gerbes have various descriptions as gauge theories and as sigma models with restrictions on nonperturbative sectors, as we shall review later. In any event, a gerbe is a special kind of stack.

2 Review of Bagger-Witten and Fayet-Iliopoulos

2.1 Four-dimensional supergravity

The Bagger-Witten line bundle in four-dimensional $N = 1$ supergravity was originally introduced in [1] in order to understand a technical issue. First, recall that across coordinate patches on the moduli space M (the target space of the scalars in the supergravity), the Kähler potential transforms as

$$K \mapsto K + f + \bar{f}.$$

In rigid supersymmetry, this is automatically a symmetry of the theory, but in supergravity, it is not, unless combined with corresponding actions [6] [(23.9)] on the gravitino ψ_μ and scalar superpartners χ^i :

$$\psi_\mu \mapsto \exp\left(-\frac{i}{2}\text{Im } f\right) \psi_\mu, \quad \chi^i \mapsto \exp\left(+\frac{i}{2}\text{Im } f\right) \chi^i.$$

(Since these are chiral rotations, there are potential anomalies, as discussed in *e.g.* [7].)

Now, let us consider² consistency across triple overlaps on the moduli space. Let $\{U_\alpha\}$ be a set of open patches on the moduli space M . On triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$, one has

$$\begin{aligned} f_{\alpha\beta} + \bar{f}_{\alpha\beta} + f_{\beta\gamma} + \bar{f}_{\beta\gamma} + f_{\gamma\alpha} + \bar{f}_{\gamma\alpha} &= (K_\beta - K_\alpha) + (K_\gamma - K_\beta) + (K_\alpha - K_\gamma), \\ &= 0. \end{aligned}$$

(Intuitively, this is the statement that the Kähler potential should return to itself.) This merely implies

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = h_{\alpha\beta\gamma} \tag{1}$$

for purely imaginary cochains $h_{\alpha\beta\gamma}$. (Since they are also holomorphic, they must be constant.) Furthermore, on a quadruple overlap $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$,

$$\begin{aligned} h_{\alpha\beta\gamma} - h_{\alpha\beta\delta} + h_{\alpha\gamma\delta} - h_{\beta\gamma\delta} \\ &= (f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}) - (f_{\alpha\beta} + f_{\beta\delta} + f_{\delta\alpha}) + (f_{\alpha\gamma} + f_{\gamma\delta} + f_{\delta\alpha}) - (f_{\beta\gamma} + f_{\gamma\delta} + f_{\delta\beta}), \\ &= 0, \end{aligned} \tag{2}$$

using the convention that $f_{\alpha\beta} = -f_{\beta\alpha}$.

² We will assume that the four-dimensional spacetime theory has exactly $N = 1$ supergravity, and not more supersymmetry. In cases with additional supersymmetry, the R symmetry group would be larger, and so across coordinate patches on the moduli space, the $U(1)_R$ we are ultimately discussing could mix with other parts of the R -symmetry group, making this story more complicated. Instead of a Bagger-Witten line bundle, one might have a Bagger-Witten $SU(2)_R$ bundle, for example.

If there is no gauge symmetry, then we can further constraint the $h_{\alpha\beta\gamma}$ using the fermions. For example, using the fact that

$$\chi_\beta^i = \exp\left(+\frac{i}{2}\text{Im } f_{\alpha\beta}\right) \chi_\alpha^i,$$

we see that

$$\begin{aligned} \chi_\gamma^i &= \exp\left(+\frac{i}{2}\text{Im } f_{\beta\gamma}\right) \chi_\beta^i = \exp\left(+\frac{i}{2}\text{Im } f_{\beta\gamma}\right) \exp\left(+\frac{i}{2}\text{Im } f_{\alpha\beta}\right) \chi_\alpha^i, \\ \text{also} &= \exp\left(+\frac{i}{2}\text{Im } f_{\alpha\gamma}\right) \chi_\alpha^i, \end{aligned} \quad (3)$$

implying that

$$\frac{i}{2}\text{Im } (f_{\beta\gamma} + f_{\alpha\beta} - f_{\alpha\gamma}) = \frac{i}{2}(-i)h_{\alpha\beta\gamma} = 2\pi i n_{\alpha\beta\gamma},$$

or more simply,

$$h_{\alpha\beta\gamma} = 4\pi i n_{\alpha\beta\gamma}, \quad (4)$$

for integers $n_{\alpha\beta\gamma}$.

In this case, $\exp(-f_{\alpha\beta}/2)$ define transition functions for an honest line bundle, and the $h_{\alpha\beta\gamma}$ define the first Chern class, by realizing the coboundary map in the long exact sequence associated to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \longrightarrow 1.$$

Furthermore, condition (4) implies that the first Chern class must be even.

However, if there is a gauge symmetry, we have more flexibility, as the fermions need only close on triple overlaps up to gauge transformations, weakening the condition (3) and leaving the $h_{\alpha\beta\gamma}$ less constrained. (In particular, in supergravity, gauge transformations will in general also act on the gravitino, see for example [6] [(25.25)].) As a result, the $h_{\alpha\beta\gamma}$ need not obey (4) in gauge theories, and so the transition functions $\exp(-f_{\alpha\beta}/2)$ need not define an honest line bundle over the moduli space in a gauge theory, as they need not close on triple overlaps:

$$\exp(-f_{\alpha\beta}/2) \exp(-f_{\beta\gamma}/2) \exp(-f_{\gamma\alpha}/2) = \exp(-h_{\alpha\beta\gamma}/2) \neq 1.$$

This is not merely an abstract consideration, but will arise explicitly when studying *e.g.* moduli spaces of elliptic curves, as we shall discuss in section 3.

Let us explore this more general case, in which $h_{\alpha\beta\gamma} \notin 4\pi i\mathbb{Z}$. Condition (2) means that the $\{h_{\alpha\beta\gamma}\}$ define a 2-cocycle, and the $\{f_{\alpha\beta}\}$ define what is known as a ‘twisted’ bundle, encountered more commonly in discussions of D-branes in B field backgrounds. Now, in general, the rank of a twisted bundle is related to the cohomology class of the twisting cocycle $\{h_{\alpha\beta\gamma}\}$. In addition, condition (1) implies that the $\{h_{\alpha\beta\gamma}\}$ also define a 2-coboundary, and

hence are trivial in cohomology, which is consistent with the fact that our ‘twisted’ bundle is rank one. The resulting structure is sometimes called a ‘fractional’ line bundle (see *e.g.* [8]).

A simple example of a fractional line bundle can be constructed as follows. Construct a projective space by quotienting $\mathbb{C}^2 - \{0\}$ by \mathbb{C}^\times with weights $(2, 2)$ rather than $(1, 1)$, and construct a line bundle by adding another \mathbb{C} factor to the quotient, of weight 1. This is the line bundle

$$\mathcal{O}(1) \longrightarrow \mathbb{P}_{[2,2]}^1,$$

which is sometimes denoted $\mathcal{O}(1/2)$. Its tensor square is the ordinary line bundle $\mathcal{O}(1)$ on \mathbb{P}^1 , but it itself is not an honest line bundle.

Regardless of whether it is an honest line bundle or merely a fractional one, we shall refer to the line bundle defined by the transition functions $\exp(-f_{\alpha\beta}/2)$ as the Bagger-Witten line bundle and denote it \mathcal{L}_{BW} .

As we shall outline later (see also *e.g.* [8] and references therein for background), if the moduli space is replaced by a gerbe over the original moduli space, such fractional line bundles can sometimes become³ honest line bundles over the new gerbe moduli ‘space’, or rather, moduli stack. Physically, sigma models on gerbes and stacks correspond to gauge theories, as discussed in *e.g.* [9–13] for two- and four-dimensional theories, precisely the circumstances just outlined in which fractional line bundles might arise in four-dimensional supergravities. See in particular [13] for further discussion of supergravity theories in which the moduli ‘space’ is a stack, and the Bagger-Witten line bundle fractional, and the implications for fractional quantizations.

Partly as a result of their alternate interpretation as honest line bundles on stacks, such fractional line bundles have properties that are very similar to ordinary line bundles. For example, given fractional line bundles $\mathcal{L}, \tilde{\mathcal{L}}$ defined by (logarithms of) transition functions $f_{\alpha\beta}, \tilde{f}_{\alpha\beta}$ on double overlaps and $h_{\alpha\beta\gamma}, \tilde{h}_{\alpha\beta\gamma}$ on triple overlaps, there is a tensor product $\mathcal{L} \otimes \tilde{\mathcal{L}}$ defined by (logarithms of) transition functions

$$f_{\alpha\beta} + \tilde{f}_{\alpha\beta}$$

on double overlaps and

$$h_{\alpha\beta\gamma} + \tilde{h}_{\alpha\beta\gamma}$$

on triple overlaps. Thus, we can manipulate fractional line bundles for some purposes as if they were honest line bundles.

³ Equivalently, a map into a gerbe is a map into the underlying space with a restriction on allowed degrees. If the lift of a fractional line bundle to a gerbe is an honest line bundle on the gerbe, then a map ϕ into the gerbe is a map ϕ into the underlying space such that $\phi^*(h_{\alpha\beta\gamma}) \in 2\pi i\mathbb{Z}$, so that the pullback by ϕ of the fractional line bundle is an honest line bundle. See *e.g.* [13] for this alternative interpretation of gerbes in terms of sigma models with restrictions on allowed nonperturbative sectors.

The gravitino is in principle a spinor-valued C^∞ section⁴ of $TX \otimes \phi^* \mathcal{L}_{\text{BW}}$, where X is the four-dimensional spacetime and $\phi : X \rightarrow M$ the scalar vevs, and the scalar superpartners are spinor-valued sections of $\phi^*(TM \otimes \mathcal{L}_{\text{BW}}^{-1})$. In this language, if the Bagger-Witten line bundle is an honest line bundle, then the Kähler form on M is⁵ a (de Rham representative of) $c_1(\mathcal{L}_{\text{BW}}^{-2})$, and so from demanding positivity of the fermion kinetic terms, if the Bagger-Witten line bundle \mathcal{L}_{BW} is an honest line bundle, then it must be a negative bundle, or more generally, $\mathcal{L}_{\text{BW}}^{-1}$ must be ample.

In this language, the superpotential W transforms as

$$W \mapsto W \exp(-f),$$

or equivalently is a meromorphic section of $\mathcal{L}_{\text{BW}}^{\otimes 2}$, so that the Yukawa couplings

$$\exp(K/2) \chi^i \chi^j D_i D_j W$$

are invariant.

At this point, we should note that fractional line bundles admit neither meromorphic sections nor higher sheaf cohomology, neither as fractional line bundles on an underlying space (or stack), nor as fractional or honest line bundles on a gerbe over the space or stack. Therefore, if $\mathcal{L}_{\text{BW}}^{\otimes 2}$ is a fractional line bundle, then as the superpotential is a meromorphic section of $\mathcal{L}_{\text{BW}}^{\otimes 2}$, it does not exist globally over the moduli space.

Four-dimensional Fayet-Iliopoulos parameters⁶ are closely interrelated [7, 13]. To gauge the action of a group G , one must specify not only how it acts on⁷ the moduli space M of scalar vevs, but also how it acts on the Bagger-Witten line bundle. Briefly, under the action of the gauge group [6] [(25.14)],

$$\begin{aligned} \delta \phi^i &= \epsilon^{(a)} X^{(a)i}, \\ \delta A_\mu^{(a)} &= \partial_\mu \epsilon^{(a)} + f^{abc} \epsilon^{(b)} A_\mu^{(c)}, \end{aligned}$$

for $X^{(a)}$ a holomorphic vector field representing the infinitesimal group action, the Kähler potential transforms as

$$\delta K = \epsilon^{(a)} F^{(a)} + \epsilon^{(a)} \overline{F}^{(a)},$$

⁴ Strictly speaking, the given expression is only derived for the case that \mathcal{L}_{BW} is an honest bundle. In that case, there is a noncanonical C^∞ isomorphism $\overline{\mathcal{L}}_{\text{BW}} \cong \mathcal{L}_{\text{BW}}^{-1}$, which we have utilized above. More generally, we should interpret the gravitino as a smooth section of $TX \otimes \phi^* \mathcal{L}_{\text{BW}}^{+1/2} \otimes \phi^* \overline{\mathcal{L}}_{\text{BW}}^{-1/2}$. Similar remarks hold for the scalar superpartners.

⁵ In type II compactifications, it was argued in [3] that the Kähler form is a de Rham representative of the first Chern class of the cubic tensor power, rather than the square.

⁶ We are referring to moduli-independent parameters. Historically, it was thought it was thought that they must vanish for reasons outlined in [14]. The paper [15] pointed out a loophole in those arguments, that they might be nonzero so long as they were quantized, which was exploited and used to link Bagger-Witten and Fayet-Iliopoulos in [7, 13].

⁷ In principle, we could work with either the compactified moduli space or omit points at infinite distance. Since G acts by isometries, points at infinite distance are merely exchanged with one another, not with points at finite distance.

where $F^{(a)} = X^{(a)}K + iD^{(a)}$ [6][(25.4)].

Thus, shifts in the imaginary part of $F^{(a)}$ correspond to Fayet-Iliopoulos parameters. Furthermore, fermions also transform: the scalar superpartners, gauginos, and gravitino transform as [6][(25.14)]

$$\begin{aligned}\delta\chi^i &= \epsilon^{(a)} \left(\frac{\partial X^{(a)i}}{\partial\phi^j} \chi^j + \frac{i}{2} \text{Im } F^{(a)} \chi^i \right), \\ \delta\lambda^{(a)} &= f^{abc} \epsilon^{(b)} \lambda^{(c)} - \frac{i}{2} \epsilon^{(a)} \text{Im } F^{(a)} \lambda^{(a)}, \\ \delta\psi_\mu &= -\frac{i}{2} \epsilon^{(a)} \text{Im } F^{(a)} \psi_\mu,\end{aligned}$$

and the terms proportional to $\text{Im } F^{(a)}$ were interpreted in [7] as describing the infinitesimal lift of the group action to the Bagger-Witten line bundle. (Strictly speaking, [7] assumed the Bagger-Witten line bundle is an honest line bundle, but effectively identical considerations apply to fractional line bundles, and result in the same counting, so we shall ignore the distinction.)

Lifts of group actions to bundles need not exist in general and, when they do exist, are not unique. Since $\text{Im } F^{(a)} = D^{(a)} + \dots$, possible lifts correspond precisely to choices of Fayet-Iliopoulos parameter [7]. Finally, in order to lift the gauge group, not just the Lie algebra of the gauge group, such lifts are quantized, and in general are counted by $\text{Hom}(G, U(1))$.

Furthermore, it was argued in [13] that the Bagger-Witten line bundle and Fayet-Iliopoulos parameters can be combined into a single object over the moduli stack⁸ $[M/G]$, namely the Bagger-Witten line bundle over the moduli stack. This is because any vector bundle on a stack $[M/G]$ is the same as a vector bundle on M with a fixed G -equivariant structure, that is, a fixed choice of G -action. Thus, to define the Bagger-Witten line bundle on $[M/G]$, we must specify precisely both the Bagger-Witten line bundle on M as well as a choice of G -action, which is to say, the Fayet-Iliopoulos parameter. Thus, Bagger-Witten and Fayet-Iliopoulos are usefully considered to be different aspects of a single whole.

Although the Bagger-Witten line bundle was originally discussed in four-dimensional $N = 1$ supergravity, it also appears much more generally in moduli spaces of SCFTs, even in SCFTs not directly related to supersymmetric compactifications to four dimensions. We shall review Bagger-Witten (and also describe Fayet-Iliopoulos) in that more general setting next.

⁸ It was also observed in [13] that in principle, the four-dimensional realization of low-energy effective sigma models on stacks can be slightly more subtle than two-dimensional realizations. That said, IR phenomena seem to be usefully captured by stacks – see *e.g.* [17][appendix B] for a discussion of four-dimensional anomalies in the language of stacks. All that said, if G is not finite, then the stack $[M/G]$ is Artin, not Deligne-Mumford, and so formal discussions of the stack $[M/G]$ may be even less pertinent to physics, as it is only currently known how to physically realize sigma models on Deligne-Mumford stacks. Nevertheless, we retain the observation above, as we think it makes an important point.

Before proceeding, we should also note that much of our description above implicitly assumes that the moduli space or stack is smooth. In fact, the moduli stack of elliptic curves that we shall discuss in section 3 is smooth. However, in $(0,2)$ theories in two dimensions, moduli spaces and stacks are often not smooth, instead exhibiting *e.g.* multicritical behavior, as we shall also discuss in that section, so $(0,2)$ theories must be treated with more care. For this reason, in this paper we will primarily focus on $(2,2)$ theories in two dimensions.

There has also been more recent work studying global aspects of Bagger-Witten line bundles, see *e.g.* [4, 5]. From a target-space supergravity perspective, it is important to bear in mind in such discussions that over the moduli space, there are loci where the four-dimensional supergravity description breaks down. We should also observe that that work has made claims stemming from a need for the Ferrara-Zumino multiplet to be well-defined; however, the transformation law for the Ferrara-Zumino multiplet in *e.g.* [16][equ'n (2.3)] resembles that of a connection on the Bagger-Witten line bundle, and so it may suffice for such arguments if the Bagger-Witten line bundle admits a flat connection, rather than necessarily be trivial. In particular, if in some particular presentation, the transition functions for the Bagger-Witten line bundle happen to be constant, then the Ferrara-Zumino multiplet would be invariant across coordinate patches, and the Bagger-Witten line bundle would admit a flat connection. That said, this property would not be preserved under general bundle isomorphisms. In any event, we will touch on these topics in section 4.

2.2 Worldsheet realizations

Consider a moduli space of SCFTs with at least one $N = 2$ symmetry algebra in either (or both) chirality. (This is a necessary condition for spacetime supersymmetry.) As one walks around loops on the moduli space of SCFTs, the theory returns to itself up to global symmetry transformations – generically, the $U(1)_R$ symmetry in the $N = 2$ algebra [2, 3].

As a result, locally the family of SCFT's transforms under the action of some principal $U(1)$ bundle over the SCFT moduli space, and if this is a chiral $U(1)$, the transition functions should correspond to a holomorphic line bundle. If the SCFT has any additional symmetries, that bundle may be twisted. For simplicity, we will assume that there are no other R symmetries that the $U(1)_R$ symmetry could mix with as one moves around the moduli space, as would happen if worldsheet supersymmetry were enhanced beyond $N = 2$. In such cases, instead of a principal $U(1)_R$ bundle, one would have a larger bundle, for which the $U(1)_R$ structure is, at best, a subbundle.

In the case that the chiral primaries in the $N = 2$ algebras admit integrally-quantized charges in conventional normalizations (another necessary condition for spacetime supersymmetry), the SCFTs admit a spectral flow operator. Specifically, in the notation and conventions of [18], in which the two supercharges in the $N = 2$ algebra have charge ± 1 with respect to the $U(1)_R$, the spectral flow operator \mathcal{U}_θ that rotates by θ , has charge $-(c/3)\theta$.

Thus, for example, a rotation from R to NS requires spectral flow $\theta = 1/2$, and the corresponding operator has charge $-c/6$. The holomorphic top form corresponds to flow by $\theta = 1$, which has charge $-c/3$, which (up to a sign) is the degree of the holomorphic top form. In any event, as a result, the spectral flow operator $\mathcal{U}_{1/2}$ couples to the Bagger-Witten line bundle \mathcal{L}_{BW} over the moduli space, and so \mathcal{U}_1 , the spacetime superpotential, and the holomorphic top form should couple to $\mathcal{L}_{\text{BW}}^{\otimes 2}$, precisely matching expectations from supergravity. (See section 6 for examples of other bundles over the SCFT moduli space.)

Ultimately the reason for the coupling of $\mathcal{U}_{1/2}$ to the Bagger-Witten line bundle \mathcal{L}_{BW} is that the spectral flow operator $\mathcal{U}_{1/2}$ is charged under the $U(1)_R$, and also implements spacetime supersymmetry on worldsheet vertex operators. Thus, when the spectral flow operator picks up a phase under the transition functions of the $U(1)$ bundle above, the scalar superpartners and gravitinos of the target-space supergravity also pick up phases, the same phases that one identifies with the Bagger-Witten line bundle in target-space supergravity.

In passing, note that the realization in SCFT is much more general than in target-space supergravity – there is no constraint on the central charge c , for example, nor is there a constraint that the SCFT be interpreted as a nonlinear sigma model. As a result, the Bagger-Witten line bundle \mathcal{L}_{BW} plays a much more general role in SCFT moduli spaces than the target-space supergravity analysis of the last section would suggest.

As a consistency check, let us consider the Yukawa couplings. In each chiral supermultiplet, the vertex operator V_χ for the fermion is obtained by applying a spectral flow operator $\mathcal{U}_{1/2}$ to the vertex operator V_ϕ for the boson, and so over the SCFT moduli space, V_χ couples to the Bagger-Witten line bundle \mathcal{L}_{BW} . For example, $\overline{27}^3$ superpotential terms are computed by worldsheet three-point couplings of the form

$$\langle V_\chi V_\phi V_\chi \rangle$$

and so we see again that W is a meromorphic section of $\mathcal{L}_{\text{BW}}^{\otimes 2}$ over the SCFT moduli space.

The discussion above suggests, correctly, that over a moduli space of complex structures on a Calabi-Yau threefold, $\mathcal{L}_{\text{BW}}^{\otimes 2}$ should be interpreted as a (possibly fractional) line bundle defined by holomorphic top-forms, since the spectral flow operator \mathcal{U}_1 that couples to $\mathcal{L}_{\text{BW}}^{\otimes 2}$ is also interpreted in terms of the holomorphic top form. We shall denote the (possibly fractional) line bundle⁹ of holomorphic top-forms over the complex structure moduli space by \mathcal{L}_H . The fiber over a point of \mathcal{L}_H over a point of the moduli space of complex structures of a Calabi-Yau n -fold is the one-dimensional vector space $H^{n,0}(X)$, where X is the Calabi-Yau corresponding to that point on the moduli space. In principle, its holonomies can be extracted from the Picard-Fuchs equation [21–23].

⁹ See [19] for a very readable introduction to Hodge theory, which encapsulates related ideas. Definition 3.13 in that reference defines Hodge bundles of any bidegree, related to the bundle of holomorphic top-forms above. The reader should note that although the entire variation of Hodge structures carries a flat connection, the filtration consists of holomorphic subbundles which need not be flat [20].

Thus, in particular, there are two (possibly fractional) line bundles of interest over moduli spaces, namely the Bagger-Witten line bundle \mathcal{L}_{BW} and the line bundle of holomorphic top-forms \mathcal{L}_{H} , related by $\mathcal{L}_{\text{H}} \cong \mathcal{L}_{\text{BW}}^{\otimes 2}$.

As an aside, the Gukov-Vafa-Witten superpotential [24] is consistent with the structure above. It has the form

$$W = \int \Omega \wedge G$$

where Ω is a holomorphic top-form on the Calabi-Yau, and G is a flux background. The expression above implies that the superpotential is a meromorphic section of the same tensor power of the Bagger-Witten line bundle as Ω over the moduli space of complex structures, and indeed, both are local sections of $\mathcal{L}_{\text{H}} \cong \mathcal{L}_{\text{BW}}^{\otimes 2}$ over the moduli space.

So far we have discussed worldsheet realizations of (possibly fractional) Bagger-Witten line bundles. In four-dimensional spacetime theories, as previously discussed, Bagger-Witten line bundles and Fayet-Iliopoulos parameters are closely interrelated. However, on the worldsheet, they necessarily become rather different. If we have a G gauge symmetry in the spacetime theory, then that is realized on the worldsheet as a G global symmetry, and in particular, in a SCFT, as a G Kac-Moody algebra at some level. However, if we have nonzero Fayet-Iliopoulos parameters in the spacetime theory, then if spacetime supersymmetry is preserved, the gauge symmetry in the target space is necessarily Higgsed, so there is no G global symmetry or Kac-Moody algebra in the worldsheet theory, or if spacetime supersymmetry is not preserved, then we do not have a perturbative vacuum and so we do not have a SCFT with integral $U(1)_R$ charges on the worldsheet. In either event, this makes Fayet-Iliopoulos parameters more difficult to understand on the worldsheet¹⁰.

3 Examples and applications

In this section, we will study the Bagger-Witten line bundle in a simple example of a moduli space of $(2, 2)$ SCFTs, namely moduli spaces of (complex structures on) elliptic curves. These would not themselves result in a compactification to a four-dimensional $N = 1$ theory, but as noted above, Bagger-Witten line bundles arise in general $N = 2$ SCFTs, not just those with $c = 9$ and integrally-charged chiral primaries. As moduli spaces of elliptic curves have been thoroughly studied in the mathematics literature, they will provide explicit examples we can understand in detail. We will first review¹¹ the mathematics, then describe the

¹⁰ There are old worldsheet-based string one-loop computations in the literature of spacetime D-terms, see *e.g.* [25–29]. Unfortunately it is not completely clear to us whether those computations are describing moduli-independent or moduli-dependent Fayet-Iliopoulos parameters, and in principle the discussion here concerns moduli-independent cases only.

¹¹ See for example [30] for more information on the mathematics of moduli spaces of elliptic curves, and [31] for an older discussion of their Picard groups.

corresponding physics. After discussing elliptic curves, we will briefly comment on higher-dimensional examples and applications.

3.1 Moduli stacks of elliptic curves

The moduli space of elliptic curves to which we often refer in the physics literature is the quotient of the upper half plane by $PSL(2, \mathbb{Z})$, and can be understood more simply as the weighted projective stack

$$\mathbb{P}_{[2,3]}^1 - \text{point},$$

where the deleted point is an ordinary non-stacky point, corresponding to a nodal elliptic curve (morally the large-complex-structure limit point) at $i\infty$. Strictly speaking, the description above replaces the quotient singularities at the corners by stack structures, so as to get a better behaved object, but at generic points is still a space. This is known technically as the ‘coarse moduli space,’ and we shall denote it, following [30], by $\mathcal{M}_{1,1}^{\text{red}}$.

Unfortunately, the coarse moduli space is not well-behaved for many applications, including ours. Consider, for example, the line bundle \mathcal{L}_H of holomorphic top-forms over the moduli space, whose square root is the Bagger-Witten line bundle. This line bundle is well-understood in the mathematics community, and is known in this context as the Hodge line bundle¹². Unfortunately, the Hodge line bundle does not exist over the moduli space $\mathcal{M}_{1,1}^{\text{red}}$ as an ordinary bundle. Of course, locally in a small patch on the moduli space, one can often construct the Hodge line bundle, but the basic issue is that in general, if one covers the space (well, the orbifold) by open sets, then on triple intersections the transition functions necessarily do not close, an example of the more general ‘fractional’ structures discussed earlier in section 2.1. In any event, we shall return to this matter momentarily.

Perhaps more importantly, it is not possible to build a ‘universal’ elliptic curve over the coarse moduli space, which is to say, an object fibered over the coarse moduli space such that the fiber at any point is the elliptic curve corresponding to that point, with the universal property that for every family of elliptic curves parametrized by any other space S , there is a unique map from S to the moduli space such that the family over S is identical to the pullback of the universal curve. This is a well-understood phenomenon mathematically, and ultimately is due to the fact that the ‘corner’ points on the coarse moduli space, the two orbifold points, describe elliptic curves with automorphisms. For example, we can build a family of elliptic curves that cannot be described in terms of a map from S^1 to the coarse moduli space: simply take the family to be a one-dimensional family of copies of one of the elliptic curves with automorphisms, along S^1 , but somewhere along S^1 , identify copies by one of the automorphisms rather than the identity. See *e.g.* [32][section 2.3] for a readable

¹² More generally [20], the Hodge line bundle is the bundle whose fibers are determinants of the holomorphic one-forms, which in the special case of elliptic curves, happens to coincide with the line bundle of holomorphic top-forms.

discussion of this issue. In any event, we shall return to questions of existence of universal objects in moduli spaces of SCFT's in section 5.1.

To solve these problems, mathematicians replace the coarse moduli space by the stacky quotient of the upper half plane by $SL(2, \mathbb{Z})$ instead of $PSL(2, \mathbb{Z})$. This quotient has a trivially-acting \mathbb{Z}_2 everywhere, which the stack remembers, and in fact is a \mathbb{Z}_2 gerbe over the coarse moduli space $\mathcal{M}_{1,1}^{\text{red}}$. This new stack, the gerbe over $\mathcal{M}_{1,1}^{\text{red}}$, is called the fine moduli space for elliptic curves, and is denoted $\mathcal{M}_{1,1}$. More explicitly, it can be described as the weighted projective stack

$$\mathbb{P}_{[4,6]}^1 - \text{point}.$$

This stack is noncompact, as it omits the point at infinity, and has Picard group \mathbb{Z}_{12} . Its compactification is denoted $\overline{\mathcal{M}}_{1,1}$, and is simply $\mathbb{P}_{[4,6]}^1$, now including the point at infinity. The Picard group of $\overline{\mathcal{M}}_{1,1}$ is simply \mathbb{Z} .

The Hodge line bundle over both $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$ is [33], [34][lemma 2.5], [30][section 5.4, 6] nontrivial, and is the generator¹³ of the Picard group in both cases. In fact, under the \mathbb{Z}_2 center of $SL(2, \mathbb{Z})$, the Hodge line bundle has nontrivial equivariant structure. (As a result, the Hodge line bundle is a ‘fractional’ line bundle on the gerbe, of the sort that would yield in principle fractional Bagger-Witten quantizations, as discussed in *e.g.* [13]. See also [8] for more information on bundles on stacks and gerbes.) It is straightforward to see how in principle this would be the case. Under the action of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}),$$

points on the complex plane are mapped as [30][section 2.3]

$$z \mapsto (c\tau + d)^{-1}z,$$

at the same time that the complex structure modulus is mapped to

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

As a result, under the center of $SL(2, \mathbb{Z})$, $z \mapsto -z$, and so a holomorphic top-form on an elliptic curve, which is proportional to dz , maps to $-dz$, and hence is odd under the action of the \mathbb{Z}_2 center.

In passing, the fact that the Hodge line bundle on $\mathcal{M}_{1,1}$ is odd under the center of $SL(2, \mathbb{Z})$ is ultimately the reason why it does not exist as a line bundle over the coarse moduli space

¹³ In particular [20], the restriction of the generator of the Picard group of $\overline{\mathcal{M}}_{1,1}$ (given by \mathbb{Z}) is the generator of the Picard group of $\mathcal{M}_{1,1}$ (given by \mathbb{Z}_{12}). Omitting the divisor at infinity quotients the Picard group by the integers, which in this case are the subgroup generated by multiplication by 12. The tensor square of the Hodge line bundle on $\mathcal{M}_{1,1}$ is the pullback to the gerbe of the generator of the Picard group of $\mathcal{M}_{1,1}^{\text{red}}$, which is \mathbb{Z}_6 .

$\mathcal{M}_{1,1}^{\text{red}}$. We can certainly define it locally over any one open patch on the moduli space, but it is ambiguous up to a sign, and so when one tries to fit together local definitions, sometimes one runs into inconsistent sign choices which cannot be removed. This is the same type of issue that prevents spinors from being defined globally on non-Spin manifolds, for example, a connection we will make more explicit shortly.

Next, let us consider the restriction of the Hodge line bundle \mathcal{L}_H on $\mathcal{M}_{1,1}$ to the complement of the orbifold points¹⁴. The moduli space has two orbifold points, the orbits of i and $\exp(i\pi/3)$ on the upper half plane. If \mathcal{L}_H denotes the Hodge line bundle, then [33] sections of $\mathcal{L}_H^{\otimes k}$ over the moduli stack are modular forms of weight k . For example, the Eisenstein series G_{2m} for $m > 1$ is a section of $\mathcal{L}_H^{\otimes 2m}$, reflected in the transformation law

$$G_{2m}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2m} G_{2m}(\tau),$$

for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}).$$

It is easy to check that Eisenstein series are invariant under the \mathbb{Z}_2 center of $SL(2, \mathbb{Z})$, corresponding to elements $\pm I_{2 \times 2}$, hence descend to $\mathcal{M}_{1,1}^{\text{red}}$. Now, G_4 vanishes at the orbit of $\exp(i\pi/3)$ and G_6 vanishes at the orbit of i , and these are their only zeroes. Hence, the meromorphic modular form G_6/G_4 has weight two, and neither zeroes nor poles away from the two orbifold points, hence it trivializes $\mathcal{L}_H^{\otimes 2}$ on the complement of the stacky points.

The Hodge line bundle \mathcal{L}_H itself is slightly different. Because it is a ‘fractional’ bundle in the language of [8], it admits no meromorphic sections over $\mathcal{M}_{1,1}$ or $\overline{\mathcal{M}}_{1,1}$, not even if one restricts to the complement of the ‘corner’ points. This is also reflected in the fact that there are no nonzero modular forms of odd degree.

We can interpret the Hodge line bundle as a spinor over the moduli space. It can be shown that [20]

$$\mathcal{L}_H^{\otimes 2} \cong \Omega_{\mathcal{M}_{1,1}}^1$$

on the uncompactified moduli stack $\mathcal{M}_{1,1}$, and similarly

$$\mathcal{L}_H^{\otimes 2} \cong \Omega_{\overline{\mathcal{M}}_{1,1}}^1[D_\infty]$$

over the compactified moduli stack $\overline{\mathcal{M}}_{1,1}$. Over $\mathcal{M}_{1,1}$, this means that \mathcal{L}_H is a square root of the canonical bundle, hence a spinor on that complex one-dimensional moduli stack, which we will see literally reflected in the worldsheet SCFT shortly.

Now, let us turn to the corresponding physics, beginning with the role of stacks. Two-dimensional nonlinear sigma models with target stacks were described in *e.g.* [9–12], and four-dimensional low-energy effective nonlinear sigma models with target stacks were discussed

¹⁴ We would like to thank R. Hain for explaining the following argument to us.

in [13]. Briefly, in ordinary sigma models, a stack is a means of encoding a gauge theory. The simplest examples are orbifolds: a nonlinear sigma model on $[X/G]$ for G finite is the same as a G -gauged nonlinear sigma model on X . In the present case, the moduli stacks of elliptic curves described above are locally (though not globally) orbifolds, and usually orbifolds by trivially-acting \mathbb{Z}_2 's. (Even though the \mathbb{Z}_2 acts trivially geometrically, both physics and stacks still detect its presence [10].)

Now, in the present case, since the stack structures are appearing on the SCFT moduli space, we need to be slightly more careful, and distinguish several possibilities. First, if the superconformal deformation parameters are constants, or equivalently maps from a point into the stack, then they are indistinguishable from coordinates on an underlying non-stacky coarse moduli space. However, if the superconformal deformation parameters are promoted to nondynamical fields, as we shall see in section 4, then their behavior is more interesting, and we see the stack structure. For example, a map from a submanifold S of spacetime into a stack $[M/G]$ is determined by

- a principal G bundle E with connection over S , and
- a G -equivariant map $E \rightarrow M$.

If G is finite, we can recognize this as the prototype of a map into a G -orbifold, as has been discussed in *e.g.* [11]. A superconformal parameter on a moduli stack that has been promoted to a nondynamical field is then defined by data of the sort above. (In the special case that S is a point, this data specializes to a map into an underlying coarse moduli space.)

If the superconformal deformation parameter were promoted to a dynamical field instead of a nondynamical one, then we would implicitly sum over data of the form above, which would have more interesting effects. For example, two-dimensional sigma models on gerbes are equivalent to sigma models on disjoint unions of ordinary spaces [12]. So long as our deformation parameters are nondynamical, we will not see that decomposition, but if they are dynamical, then summing over the defining data above will yield the disjoint union of spaces described in [12].

Physically, over the moduli space $\mathcal{M}_{1,1}$, the operator \mathcal{U}_1 is, in principle, locally a section of the Hodge line bundle \mathcal{L}_H of holomorphic top-forms over the moduli space. However, as just outlined, the Hodge line bundle admits no meromorphic sections over the moduli space, hence the operator \mathcal{U}_1 is defined only locally, not globally on the moduli space. Specifically, it is locally a section of a fractional line bundle, and so its transition functions only close on triple overlaps up to higher cocycles.

In the worldsheet SCFT, the holomorphic top-form corresponds to an R sector vacuum – literally, a spinor on the moduli space, as we have seen mathematically. The failure of the Hodge line bundle to exist as an honest line bundle over the moduli space is precisely due

to a sign ambiguity in defining the R sector vacuum across different patches on the moduli space. Furthermore, though we have only discussed that sign ambiguity for moduli spaces of elliptic curves, it appears to be generic in $N = 2$ SCFTs, suggesting that the Hodge line bundle will typically be fractional over moduli spaces of other $N = 2$ SCFTs.

In principle the Bagger-Witten line bundle is a square root of the Hodge line bundle, but as the Hodge line bundle generates the order-12 Picard group of $\mathcal{M}_{1,1}$, it has no square root on $\mathcal{M}_{1,1}$, so to define the Bagger-Witten line bundle, we will need to replace the moduli stack $\mathcal{M}_{1,1}$.

With this motivation, we can now define our next set of stacks, which will be \mathbb{Z}_2 gerbes over $\mathcal{M}_{1,1}$, $\overline{\mathcal{M}}_{1,1}$ on which the Hodge line bundle admits a square root. In fact, tautologically¹⁵, for any line bundle L on any stack \mathfrak{X} , there is a canonically-defined \mathbb{Z}_2 gerbe on \mathfrak{X} such that the pullback of L to this gerbe has a square root. The class of the gerbe is the image of $[L]$ in $H^1(\mathfrak{X}, \mathcal{O}^\times)$ under the coboundary map for the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{O}^\times \longrightarrow \mathcal{O}^\times \longrightarrow 1.$$

We therefore define the ‘stringy’ moduli stacks $\mathcal{M}_{1,1}^S$, $\overline{\mathcal{M}}_{1,1}^S$ to be the (canonically-defined) \mathbb{Z}_2 gerbes over $\mathcal{M}_{1,1}$, $\overline{\mathcal{M}}_{1,1}$, respectively, such that the Hodge line bundle admits a square root, namely the Bagger-Witten line bundle \mathcal{L}_{BW} . This stack can alternatively be presented as [20] the (stacky) quotient of the upper half plane by $Mp(2, \mathbb{Z})$, the metaplectic¹⁶ group. It can be shown [20] that $\mathcal{M}_{1,1}^S$ is a \mathbb{Z}_4 gerbe over $\mathcal{M}_{1,1}^{\text{red}}$, that the Picard group of $\mathcal{M}_{1,1}^S$ is \mathbb{Z}_{24} , and that the Bagger-Witten line bundle \mathcal{L}_{BW} generates that Picard group. (For completeness, the Picard group of the compactification $\overline{\mathcal{M}}_{1,1}^S$ is \mathbb{Z} , and the restriction of the generator of \mathbb{Z} to the uncompactified stack $\mathcal{M}_{1,1}^S$ is the Bagger-Witten line bundle.)

¹⁵ We would like to thank T. Pantev for pointing out this construction.

¹⁶ This is the unique nontrivial central extension of $SL(2, \mathbb{Z})$ by \mathbb{Z}_2 . Its elements can be written in the form

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \pm \sqrt{c\tau + d} \right),$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}),$$

and $\sqrt{c\tau + d}$ is considered as a holomorphic function of τ in the upper half plane. The multiplication is defined as

$$(A, f(\cdot))(B, g(\cdot)) = (AB, f(B(\cdot))g(\cdot)).$$

Furthermore, it can be shown that the metaplectic group is an extension of $PSL(2, \mathbb{Z})$ by \mathbb{Z}_4 ,

$$1 \longrightarrow \mathbb{Z}_4 \longrightarrow Mp(2, \mathbb{Z}) \longrightarrow PSL(2, \mathbb{Z}) \longrightarrow 1,$$

as the two \mathbb{Z}_2 ’s arising in the center and extension of $SL(2, \mathbb{Z})$ do not commute with one another, and this is the reason for the \mathbb{Z}_4 -gerbe structure mentioned above.

In more elementary language, under the action of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}),$$

the holomorphic top-form on an elliptic curve transforms as

$$dz \mapsto (c\tau + d)^{-1} dz,$$

hence the spectral flow operator $\mathcal{U}_{1/2}$, which transforms as \sqrt{dz} , must transform under $SL(2, \mathbb{Z})$ as

$$\mathcal{U}_{1/2} \mapsto \pm \frac{1}{\sqrt{c\tau + d}} \mathcal{U}_{1/2}.$$

As a result, the spectral flow operator $\mathcal{U}_{1/2}$ is well-defined only over the stacky quotient of the upper half plane by $Mp(2, \mathbb{Z})$, which is $\mathcal{M}_{1,1}^S$.

As another consistency check, one can show that $\mathcal{L}_{\text{BW}}^{-1}$ is ample, which (as mentioned earlier) is necessary for positivity of the fermion kinetic terms in general. We can understand this as follows. For (Deligne-Mumford) stacks [20], a line bundle is ample if and only if some power is a pullback of an ample bundle from the underlying coarse moduli stack. In the present case, the generator of the Picard group on the coarse moduli space $\mathcal{M}_{1,1}^{\text{red}}$ is ample, hence $\mathcal{L}_{\text{BW}}^{-1}$ on $\mathcal{M}_{1,1}^S$ is ample, as its fourth tensor power is the pullback of an ample line bundle on $\mathcal{M}_{1,1}^{\text{red}}$.

We conjecture that the stacks $\mathcal{M}_{1,1}^S, \overline{\mathcal{M}}_{1,1}^S$ are fine moduli stacks (for the complex structure moduli) for the superconformal field theory (thus the S superscript). Of course, we do not know how to define a ‘universal SCFT’ over such a moduli stack (though see section 5.1 for some general observations.) In any event, let us at least extend this conjecture to include Kähler moduli, in a way consistent with T-duality.

As is well-known, the moduli space of $N = 2$ SCFT’s describing strings on elliptic curves can be described approximately as $O(2, 2; \mathbb{Z}) \backslash O(2, 2) / (O(2) \times O(2))$. Briefly, the $O(2, 2)$ decomposes, and as a result this can be written as, modulo \mathbb{Z}_2 ’s, a product of two copies of $PSL(2, \mathbb{Z}) \backslash SL(2) / O(2)$, which can be identified¹⁷ with two copies of the coarse moduli

¹⁷ One maps $SL(2)/U(1)$ into the upper half plane by mapping

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R}) \mapsto \frac{ai + b}{ci + d}.$$

Note that

$$\text{Im} \frac{ai + b}{ci + d} = \frac{ad - bc}{c^2 + d^2} = \frac{1}{c^2 + d^2} > 0,$$

so the image lies in the upper half plane. Moreover, the $U(1)$ coset acts as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} a \cos \theta - b \sin \theta & a \sin \theta + b \cos \theta \\ c \cos \theta - d \sin \theta & c \sin \theta + d \cos \theta \end{bmatrix},$$

space of an elliptic curve. One copy corresponds to the complex structure moduli, the other to the Kähler moduli.

If we let τ denote the complex structure parameter and σ the Kähler structure parameter, then being more careful about finite group quotients, the coarse moduli space of SCFT's is of the form [35–37]

$$(\mathcal{M}_{1,1}^{\text{red}} \times \mathcal{M}_{1,1}^{\text{red}}) / \mathbb{Z}_2 \times \mathbb{Z}_2$$

The resulting moduli space contains quotient singularities along complex codimension one subvarieties, corresponding to where either σ or τ is i or $\exp(\pi i/3)$, as well as two points of maximal enhanced symmetry ($SU(3)$, $SU(2) \times SU(2)$), where $(\sigma, \tau) = (\exp(\pi i/3), \exp(\pi i/3))$, (i, i) , lying at the intersection of the singular subvarieties. In this spirit, it is natural to conjecture that the complete fine moduli space of $N = 2$ SCFTs on elliptic curves is of a similar form, a quotient of a product of two copies of $\mathcal{M}_{1,1}^S$, one copy for complex moduli, another for Kähler moduli. However, as discussed in section 5.1, at the moment we do not have a precise definition of a ‘universal SCFT,’ so we leave this conjecture for future work.

3.2 Outline of higher dimensions

In higher dimensions, less is known in general. For example, fine moduli spaces of complex structures may or may not be gerbes over coarse moduli spaces – it is not clear whether the gerbe structure we have seen for elliptic curves will generalize. For an abelian variety of dimension n (*i.e.* a torus T^{2n}) [20], the \mathbb{Z}_2 that acted nontrivially on holomorphic one-forms on elliptic curves, acts by $(-)^n$ on holomorphic n -forms. One might expect that the fine moduli space of complex structures may have a \mathbb{Z}_2 gerbe structure, but for n even, the line bundle of holomorphic top-forms may be an honest bundle, whereas for n odd, it may be fractional.

That said, one particularly pertinent reference is [38]. This paper discussed old work [39, 40] in which it had been argued that (0,2) worldsheet moduli were generically lifted by worldsheet instanton corrections, via a study of a single worldsheet instanton. Their computations could not be translated to topological field theory computations, and so were difficult to analyze directly. The paper [38] argued that in many cases, multiple worldsheet

and so its image under the map above is

$$\frac{ae^{-i\theta}i + be^{-i\theta}}{ce^{-i\theta}i + de^{-i\theta}} = \frac{ai + b}{ci + d},$$

hence the map descends to $SL(2, \mathbb{R})/U(1)$, and in fact, it is easy to see that it also descends to $SL(2, \mathbb{R})/O(2)$. Furthermore, it is straightforward to check that this map defines an isomorphism

$$SL(2, \mathbb{R})/O(2) \cong PSL(2, \mathbb{R})/U(1) \cong \text{upper half plane}.$$

See [35] for more information.

instanton contributions will cancel out, leaving the (0,2) moduli. There has since been a significant number of papers on the subject, see *e.g.* [41–47] and references therein for a sample.

In any event, the paper [38] studied worldsheet instanton corrections in heterotic strings by working on the target space, and using the fact that the four-dimensional spacetime $N = 1$ superpotential is a section¹⁸ of $\mathcal{L}_H \cong \mathcal{L}_{BW}^{\otimes 2}$ over the compactified SCFT moduli space.

In essence, the authors of [38] argued that the spacetime superpotential for neutral moduli should vanish, and hence those moduli are unobstructed. Interestingly, as a consistency check they also studied the superpotential for charged matter couplings, which do not vanish and in fact encode, on the (2,2) locus, Gromov-Witten invariants. Those contributions diverge at singular loci on the moduli space, reflecting divergences in the worldsheet instanton sums, which in the present context are interpreted as reflecting the fact that $\mathcal{L}_{BW}^{\otimes 2}$ is a negative line bundle, hence any nonzero (meromorphic) section must have poles. On the (2,2) locus, the restriction of the Bagger-Witten line bundle to the complement of those singular loci might be trivial, but over the compactified moduli space, its structure is tied by the arguments of [38] to Gromov-Witten theory.

It is tempting to use expressions for holomorphic top-forms to compute the Bagger-Witten line bundle and the line bundle of holomorphic top-forms over compactified moduli spaces, but this does not quite work. For completeness, let us describe both the idea and how this fails. First, let us explain what we mean by expressions for holomorphic top-forms. For example, for elliptic curves constructed as degree-three polynomials p in \mathbb{P}^2 , in a patch with affine coordinates x, y on \mathbb{P}^2 , the holomorphic top-form Ω is

$$\Omega \propto \frac{dx}{\partial p / \partial y}. \quad (5)$$

Now, the coefficients of p act as homogeneous coordinates on the (coarse) moduli space of complex structures, which can be constructed by quotienting symmetries. It is tempting to argue that since Ω is of negative degree in those coefficients, it must therefore couple to a (fractional) line bundle, of negative degree if it is a line bundle, which would be consistent with earlier observations. Unfortunately¹⁹, the expression above could be multiplied by a meromorphic section of a different line bundle over the compactified moduli space, one with zeroes and poles at singular and limiting points, resulting in an Ω that is equivalent over the smooth part of the moduli space but which couples to a different (fractional) bundle over the compactified moduli space. Put another way, to uniquely specify the line bundle

¹⁸ In passing, there is a technical subtlety here which was unknown at the time [38] was written, namely that if \mathcal{L}_H is a fractional line bundle, then the superpotential cannot exist globally as \mathcal{L}_H admits no meromorphic sections. At best, one could only write down superpotentials in local patches on the moduli space. Globally, there would be no unambiguous way to assign a superpotential to every point on the moduli space – one would have different choices related by phases.

¹⁹ We would like to thank D. Morrison for explaining this matter to us.

of holomorphic top-forms over the moduli space requires a specification of its behavior near limiting points, and expressions such as equation (5) make implicit assumptions and so do not provide a unique answer.

In passing, we should also observe that existence of finite nonzero volumes of CFT moduli spaces computed in *e.g.* [48,49] would appear to suggest that over compactified moduli spaces, Bagger-Witten line bundles may be nontrivial.

4 Constraints implied by worldsheet metric

Recently the paper [5] used worldsheet arguments to constrain the Bagger-Witten line bundle. They used a variation of superconformal perturbation theory, in which the deformation parameters are promoted to nondynamical fields, and derived a constraint on the Bagger-Witten line bundle from the observation that the worldsheet Liouville field has no symmetry transformations and should be invariant across open patches on the moduli space. In this section we will carefully work through those arguments and see how they imply that the (possibly fractional) Bagger-Witten line bundle should admit a flat connection.

For any fixed initial SCFT, the paper [5] performed superconformal perturbation theory, but with the parameters promoted to non-dynamical functions on the worldsheet. This enabled them to analyze perturbation theory as more nearly a low-energy effective field theory. Since the fields are non-dynamical, they can be restricted to any open patch on the SCFT moduli space. (See also *e.g.* [50,51] for other references that also promote the superconformal deformation parameters to non-dynamical fields.)

For simplicity, for the moment we shall assume (2,2) worldsheet supersymmetry. (We shall comment on subtleties in the more general (0,2) case later.) In a (2,2) supersymmetric theory, it is most natural for one of the complex and Kähler moduli to be described by ordinary chiral multiplets, and the other to be described by twisted chiral multiplets. The result is a moduli space with a generalized complex structure in Hitchin’s sense. An example of the resulting structure is implicit in the ‘ur-theory’ described implicitly in [52].

There is a universal contribution involving a supersymmetric multiplet denoted Σ . In superconformal gauge, its bosonic part has the form [5][appendix C.1]

$$\Sigma = \sigma + ia + \cdots,$$

where σ is the conformal factor in the worldsheet metric and a is defined by a non-dynamical $U(1)$ gauge field A_μ over the moduli space (coupling to the Bagger-Witten line bundle), as

$$A_\mu = \epsilon_{\mu\nu} \partial^\nu a.$$

(This definition of a implicitly forces constraints on the $U(1)_R$ bundle over the moduli space, as we shall discuss momentarily.) They derive that the correct kinetic term for Σ , which

encodes the supersymmetrization of the conformal anomaly, has the form (in superconformal gauge) [5][equ'n (3.11)]

$$\int d^2x \int d^4\theta \left(\Sigma + \bar{\Sigma} - \frac{6}{c}K \right)^2 \quad (6)$$

where K is the Kähler potential on (that part of) the moduli space.

The action (6) above encodes a kinetic term for Σ , namely

$$\int d^4\theta \bar{\Sigma} \Sigma,$$

as well as counterterms that arise when transforming either Σ or K , such as

$$\int d^4\theta \bar{\Sigma} F.$$

In particular, since $\bar{D}^2 \bar{\Sigma} = \mathcal{R}$, the chiral curvature superfield, the line above equals [5][equ'n (3.3)]

$$\int d^2\theta \mathcal{R} F,$$

which was used in [5] to motivate the action (6).

Manifestly, the action (6) is formally invariant under

$$K \mapsto K + F + \bar{F}, \quad \Sigma \mapsto \Sigma + \frac{6}{c}F,$$

for any chiral superfield F . However, this would also modify the worldsheet metric, as it would shift σ , and the lack of gauge transformations of σ plays a crucial role.

Let us study more closely the allowed transformations of Σ . Consider a gauge transformation of the background $U(1)$ gauge field (as would *a priori* happen across open patches on the moduli space)

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda.$$

Define $\tilde{\Lambda}$ by,

$$\partial_\mu \Lambda = \epsilon_{\mu\nu} \partial^\nu \tilde{\Lambda},$$

or in components,

$$\begin{aligned} \partial_1 \Lambda &= \partial^2 \tilde{\Lambda}, \\ \partial_2 \Lambda &= -\partial^1 \tilde{\Lambda}, \end{aligned}$$

which the reader will recognize as being more-or-less the Cauchy-Riemann equations. Note that existence of $\tilde{\Lambda}$ implies that

$$\partial_\mu \partial^\mu \Lambda = 0,$$

hence

$$\Lambda = f(z) + f(z)^*,$$

(using both the fact that it is real and it solves Laplace's equation,) so the field a is only defined globally (as a section of an affine bundle) if the original $U(1)$ bundle over the moduli space, to which A_μ couples, is determined by a holomorphic bundle. As the Bagger-Witten line bundle is indeed a holomorphic line bundle over the moduli space, this constraint will always be satisfied. Then, for the Λ above,

$$\tilde{\Lambda} = \text{Im}(f(z) - f(z)^*)$$

and under the gauge transformation above, $a \mapsto a + \tilde{\Lambda}$. In particular, if A_μ coupled to a $U(1)_V$ transformation, then a undergoes a $U(1)_A$ transformation, and vice-versa.

Now that we have examined the allowed gauge transformations, let us describe how these fit together systematically. For the moment, we will only work on open covers of the moduli space, as SCFT perturbation theory can certainly describe open covers; in section 5.1, we will discuss potential obstructions to building a ‘universal SCFT,’ with parameters ranging over the entire moduli space. Let U_α be an open cover of the smooth part of the moduli space. (For reasons we shall discuss later, we exclude singular points from the discussion.) Let Σ_α denote the Σ field over U_α , and K_α the Kähler potential across U_α .

We can glue these fields together across coordinate patches via field redefinitions. Since $\sigma \propto \Sigma + \bar{\Sigma}$ does not admit transformations across coordinate patches, we need to find a set of field redefinitions

$$\Sigma_\alpha \mapsto \Sigma_\alpha + C_\alpha,$$

for superfields C_α on each patch, such that on every overlap $U_\alpha \cap U_\beta$ on the moduli space,

$$(\Sigma_\alpha + \bar{\Sigma}_\alpha + C_\alpha + \bar{C}_\alpha)|_{U_\alpha \cap U_\beta} = (\Sigma_\beta + \bar{\Sigma}_\beta + C_\beta + \bar{C}_\beta)|_{U_\alpha \cap U_\beta}. \quad (7)$$

The Liouville field defined by

$$\Sigma_\alpha + \bar{\Sigma}_\alpha + C_\alpha + \bar{C}_\alpha \quad (8)$$

on each patch on the moduli space will then be globally well-defined, and we can then talk about a globally-defined “ $\Sigma + \bar{\Sigma}$ ” whose restriction to open patch U_α is the sum (8) above.

As previously remarked, on an overlap $U_\alpha \cap U_\beta$, the action (6) is invariant under

$$K_\alpha \mapsto K_\alpha + F_{\alpha\beta} + \bar{F}_{\alpha\beta}, \quad \Sigma_\alpha \mapsto \Sigma_\alpha + \frac{6}{c} F_{\alpha\beta},$$

where $F_{\alpha\beta}$ is a non-dynamical chiral superfield on $U_\alpha \cap U_\beta$ on the moduli space whose bosonic components are the (logarithms of the) transition functions $f_{\alpha\beta}$ of the Bagger-Witten line bundle.

Suppose, for example, that

$$F_{\alpha\beta} = C_\alpha|_{U_\alpha \cap U_\beta} - C_\beta|_{U_\alpha \cap U_\beta},$$

then $F_{\alpha\beta}$ can be absorbed into field redefinitions

$$\Sigma_\alpha \mapsto \Sigma'_\alpha \equiv \Sigma_\alpha + \frac{6}{c}C_\alpha, \quad \Sigma_\beta \mapsto \Sigma'_\beta \equiv \Sigma_\beta + \frac{6}{c}C_\beta,$$

so that after field redefinitions, the Σ' s are invariant across overlaps:

$$\Sigma'_\alpha|_{U_\alpha \cap U_\beta} = \left(\Sigma_\alpha + \frac{6}{c}C_\alpha \right) \Big|_{U_\alpha \cap U_\beta} \mapsto \Sigma'_\beta|_{U_\alpha \cap U_\beta} = \left(\Sigma_\beta + \frac{6}{c}C_\beta \right) \Big|_{U_\alpha \cap U_\beta}.$$

In this case, the Bagger-Witten line bundle would be holomorphically trivial.

However, a weaker condition will suffice to leave the action (6) invariant. Suppose instead that

$$F_{\alpha\beta} = \tilde{F}_{\alpha\beta} + C_\alpha|_{U_\alpha \cap U_\beta} - C_\beta|_{U_\alpha \cap U_\beta},$$

where $\tilde{F}_{\alpha\beta}$ is pure imaginary (and hence constant). In this case, the C_α 's can be absorbed into field redefinitions of the Σ_α 's, as

$$\Sigma_\alpha \mapsto \Sigma'_\alpha \equiv \Sigma_\alpha + \frac{6}{c}C_\alpha, \quad \Sigma_\beta \mapsto \Sigma'_\beta \equiv \Sigma_\beta + \frac{6}{c}C_\beta,$$

so that on overlaps,

$$\Sigma'_\alpha|_{U_\alpha \cap U_\beta} \mapsto \tilde{F}_{\alpha\beta} + \Sigma'_\beta|_{U_\alpha \cap U_\beta}.$$

Although Σ' is no longer invariant across overlaps, the sum $\Sigma' + \overline{\Sigma'}$ is invariant, satisfying the condition (7) and so the Liouville field is still well-defined globally. This is clearly the general case consistent with making the Liouville field well-defined globally, and this only requires that the Bagger-Witten line bundle admit a flat connection, but not necessarily be trivial.

At this point, let us make two observations.

- First, the Bagger-Witten line bundle \mathcal{L}_{BW} , with transition functions $\exp(-f_{\alpha\beta}/2)$, might not be an honest line bundle. We have described a constraint on the $f_{\alpha\beta}$ on the double overlaps $U_{\alpha\beta}$, but we have not given a constraint on the cocycles $h_{\alpha\beta\gamma}$ on triple overlaps beyond being pure imaginary, the only constraint derived earlier. If those cocycles $h_{\alpha\beta\gamma} \notin 2\pi i\mathbb{Z}$, then \mathcal{L}_{BW} will be fractional.
- Over the moduli space, $\exp(\Sigma_\alpha)$ transforms as a local section of $\mathcal{L}_{\text{BW}}^{-6/c}$. In general, depending upon \mathcal{L}_{BW} and the ratio $6/c$, this may be a fractional line bundle. If that should be the case, then $\exp(\Sigma_\alpha)$ can only be defined locally on the moduli space – the worldsheet metric will be well-defined, but the imaginary part of Σ will only be defined in patches.

So far we have implicitly assumed that the moduli space over which we are working is smooth, or rather that we are working on the complement of singular points or points at infinite distance. (In particular, in SCFT's obtained from Calabi-Yau's, there will more or less always be singularities and (large-radius) points at infinite distance.) However, formally we would think the argument above should also apply to smooth Deligne-Mumford stacks, following the prescription in [13], as such stacks can be covered by smooth open patches on which one can perform differential geometry in the usual fashion.

Let us compare the claim above to results for moduli spaces of elliptic curves, discussed in section 3. There, we saw that over the uncompactified moduli space, the Bagger-Witten line bundle is nontrivial (and fractional), but torsion, satisfying the constraint above. If we excise the two points of enhanced stabilizer from the uncompactified moduli space, the situation is similar: the restriction of $\mathcal{L}_{\text{BW}}^{\otimes 4} \cong \mathcal{L}_{\text{H}}^{\otimes 2}$ is trivializable, as discussed in section 3, but as the Bagger-Witten line bundle \mathcal{L}_{BW} is not an honest line bundle, we cannot define a section, not even over this excised region, and so it is not quite correct to say that its restriction is trivial.

In passing, in recent discussions of supersymmetric localization, it has been argued that the S^2 partition function $Z \propto \exp(-K)$. Clearly, this transforms over the moduli space as a C^∞ section of

$$\mathcal{L}_{\text{H}} \otimes \overline{\mathcal{L}}_{\text{H}} = \mathcal{L}_{\text{BW}}^{\otimes 2} \otimes \overline{\mathcal{L}}_{\text{BW}}^{\otimes 2}.$$

However, under the circumstances we have discussed here, the transition functions of \mathcal{L}_{BW} can be chosen to be pure imaginary, so that $\overline{\mathcal{L}}_{\text{BW}} \cong \mathcal{L}_{\text{BW}}^{-1}$, and so the line bundles above cancel out, making $\exp(-K)$ a function over the moduli space (after suitable redefinitions on local patches).

For more or less the same reasons, the left-right symmetric spectral flow operator should be unambiguously defined. This is pleasant²⁰ as the operator shows up in diagonal modular invariants of (2,2) SCFTs, and corresponds to the unique NS-NS ground state.

So far we have focused on (2,2) theories. In (0,2) theories, there is an additional subtlety, namely the existence of multicritical points, *i.e.* subvarieties connecting sections of the moduli space of different dimension, joined at well-behaved SCFT's. This cannot happen in (2,2) theories, as it would require the chiral ring to jump, and the chiral ring is well-known to have the same dimension across the (2,2) SCFT moduli space. However, in (0,2) theories, it is much more common. For example, if we take the (2,2) quintic and add suitable left-moving fields, the resulting (0,2) theory gains a $SO(10)$ branch.

²⁰ We would like to thank I. Melnikov for making this observation.

5 Universal structures over moduli spaces of SCFTs

5.1 Potential obstructions to existence of universal SCFTs

It would be natural to try to interpret the results of section 4 in terms of some sort of ‘universal SCFT,’ a physical theory in which superconformal deformation parameters have been promoted to non-dynamical fields covering the entire moduli space or stack, such that one gets a unique ordinary SCFT at any point of the moduli stack by ‘freezing’ the superconformal deformation parameters to an appropriate value. In QFT terms, we do not know a precise definition of such a structure. However, in mathematics, such structures are often studied, and constraints on the existence of such an object are well-known.

In this section, we will apply some of the general principles known in mathematics to give a brief overview of possible obstructions to the existence of such a hypothetical universal SCFT structure, and associated issues.

In general, in constructing a moduli space of objects of some type, it sometimes happens that to a single point on the moduli space can be associated multiple objects. For example, in moduli spaces of holomorphic vector bundles, there can be loci along which several semistable bundles are associated to the same point on the moduli space (see *e.g.* [53–55] and references therein). This issue arises in heterotic strings; however, of the semistable bundles over any one point, only one defines a SCFT, the others define massive theories. In principle, analogues also exist at singular limiting points in moduli spaces of complex structures, see *e.g.* [56].

If a single point on the moduli space can be associated to multiple objects, then, it is not possible to construct a universal object over the moduli space. However, intuitively this only happens at singular points on the moduli space, so if we restrict to smooth points, it seems plausible that we can avoid this issue.

In principle, another issue that can arise involves objects with additional automorphisms. If one walks around the moduli space and returns to oneself but only up to an automorphism, it may not be possible to find a universal object. One way to resolve this issue is to change the moduli problem by adding extra data that removes the automorphisms, hence ‘rigidifying’ the problem. A different approach that does not involve changing the objects themselves is to replace moduli spaces with moduli stacks, in which the automorphisms are essentially divided out. We have speculated about such stacks in the case of moduli spaces of elliptic curves in section 3; however, we should also emphasize that, due to the fact that SCFTs can have in principle many subtle symmetries, including higher group symmetries (see *e.g.* [57]), it would be difficult to completely settle this issue for moduli spaces of SCFTs at this time.

One of the subtleties we see in the example of moduli spaces of elliptic curves in section 3

is that even if a universal SCFT exists, it may not be possible to construct operators that extend globally across the moduli space. For example, as discussed in section 3, the spectral flow operator $\mathcal{U}_{1/2}$ is locally a section of the Bagger-Witten line bundle over a gerbe over the moduli space of elliptic curves. However, the Bagger-Witten line bundle is not an honest line bundle, it is only a fractional line bundle, and as such, admits no meromorphic sections, neither over the stack nor over the underlying coarse moduli space. Even if we omit the points with enhanced stabilizers, the restriction of the Bagger-Witten line bundle to the complement is still fractional and still does not admit meromorphic sections, short of performing some noncanonical transformation to remove the twisting. As a result, because there is no global meromorphic section, it is not possible to unambiguously associate an operator $\mathcal{U}_{1/2}$ to every point of the moduli space – we can make local choices, but there does not exist a global consistent choice.

In passing, it is tempting to apply ideas about existence of universal structures to F theory compactifications, to argue for the necessity of non-mutually-local branes. Consider an F theory compactification to six dimensions, on $T^2 \times K3$, along the fibers of an elliptically-fibered $K3$. Morally the \mathbb{P}^1 base of the $K3$ is parameterizing a family of elliptic curves, which (because we have compactified down to six dimensions), we can think of has a family of $N = 2$ SCFTs. If a universal SCFT structure existed over that entire parameter space, that fact might suggest that the fibration should be trivial, admitting no monodromies. However, the existence of non-mutually-local branes means that we do not have a global family of SCFTs, and so such an argument does not apply. In general terms, this appears to be broadly consistent with observations about monodromies resulting from non-mutually-local branes in [58][section 6].

Given that building a well-behaved moduli space of elliptic curves requires working with stacks instead of spaces, one might wonder whether stacks suffice when describing universal objects over moduli stacks of SCFTs. One possibility is that one might have to work with higher stacks, a suggestion which is implicit in the proposed construction we outline next.

5.2 Sheaves of RG flows

In passing, let us very briefly outline a proposal for another approach to constructing possible ‘universal SCFT’s.’

Cover the CFT moduli space by open patches $\{U_\alpha\}$. To each patch U_α , associate a family of quantum field theories (not necessarily conformal field theories), in which deformation parameters have been promoted to non-dynamical fields, and which in principle should RG flow to conformal field theories associated with points on U_α . To ‘glue’ these together on overlaps $U_\alpha \cap U_\beta$, we require that the two quantum field theories RG flow to the same theory at some scale $\Lambda_{\alpha\beta}$, modulo field redefinitions and irrelevant operators. On triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$, there is no guarantee that the three scales $\Lambda_{\alpha\beta}$, $\Lambda_{\beta\gamma}$, $\Lambda_{\gamma\alpha}$ or their associated

field redefinitions or irrelevant operator insertions will all be comparable, so we must instead require that on triple overlaps, the three quantum field theories associated to the three patches will all RG flow to the same theory at some scale $\Lambda_{\alpha\beta\gamma}$, modulo field redefinitions and irrelevant operators. One would of necessity continue to impose the same constraint on quadruple and all higher overlaps.

In broad brushstrokes, by associating multiple different quantum field theories to each open set, one would get a type of infinity sheaf in this fashion, with gluing conditions provided by renormalization group flow (and field redefinitions and irrelevant operators) on intersections. A potential ‘universal SCFT’ would be a global section of that sheaf.

In this language, the idea that the spectral flow operator $\mathcal{U}_{1/2}$ is a section of a fractional line bundle, is tied to the idea that under renormalization group flow, a UV R symmetry may mix with non- R symmetries, as has been applied recently in *e.g.* c -extremization [59]. Intuitively, the IR R symmetries in different patches on the moduli space might subtly differ if there are other symmetries in the theory, and in particular could fail to close on triple overlaps.

6 Left-moving analogues in heterotic compactifications

So far in this paper we have primarily focused on Bagger-Witten structures in (2,2) supersymmetric worldsheet SCFTs. In this section we will briefly outline how structures analogous to the Bagger-Witten line bundle can arise on moduli spaces of (0,2) SCFTs.

In a typical perturbative heterotic compactification on a Calabi-Yau, involving an $SU(n)$ gauge bundle, the left-moving fermions also admit a $U(1)$ symmetry (which becomes²¹ the left-moving $U(1)_R$ on the (2,2) locus), which rotates the left-moving fermions by phases. One might ask about the analogue of Bagger-Witten for this symmetry. In fact, that left-moving $U(1)$ forms part of the low-energy gauge symmetry. For example, in a compactification of an $E_8 \times E_8$ string on a Calabi-Yau threefold with an $SU(3)$ bundle, the first E_8 is broken to an E_6 , which is realized on the worldsheet via a $\text{Spin}(10) \times U(1)$ subalgebra. The $U(1)$ factor is precisely the left-moving $U(1)$ described above. In any event, as one walks around the SCFT moduli space, the left-movers come back to themselves up to $\text{Spin}(10) \times U(1)$ and, ultimately, E_6 rotations. Thus, if one has low-energy gauge symmetry G , then over the SCFT moduli space, target-space fields behave as sections of a vector bundle associated to a principal G bundle, of which the $U(1)$ above defines a rank one subbundle.

Part of the reason for our focus on (2,2) theories is that, as explained previously in section 4, moduli spaces of (0,2) SCFTs have multicritical points at which varieties of different

²¹ Just as ordinary mirror symmetry flips the sign of charges under the left-moving $U(1)_R$, (0,2) mirror symmetry is similarly defined by a sign flip in charges under this canonical $U(1)$.

dimension intersect one another, and so are not amenable to analyses which assume the moduli space is smooth. Our observations above, however, do not require the moduli space to be smooth.

7 Conclusions

In this paper we have examined recent claims about the Bagger-Witten line bundle in the context of a specific concrete example, namely the moduli space of elliptic curves. We have shown how in general the Bagger-Witten line bundle is not actually an honest line bundle, but rather will only be ‘fractional,’ as in fact happens over moduli spaces of elliptic curves. As a fractional line bundle, it is nontrivial but torsion, which is consistent with recent results of [5] stipulating that consistency of the worldsheet metric in families of SCFTs requires that the Bagger-Witten line bundle admit a flat connection. We have also examined more general features of such ‘universal’ constructions of families of SCFTs in two dimensions, and proposed another construction utilizing sheaves of massive quantum field theories. Finally, we have discussed other bundles appearing over moduli spaces of SCFTs.

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